Hydrodynamic Turbulence: a 19th Century Problem
with a challenge for the 21st Century

Victor L'vov and Itamar Procaccia

Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel

I. TURBULENCE FOR THE PHYSICIST AND FOR THE ENGINEER

Sir Horace Lamb once said "I am an old man now, and when I die and go to Heaven there are two matters on which I hope enlightenment. One is quantum electro-dynamics and the other is turbulence of fluids. About the former, I am really rather optimistic" [1]. Possibly Lamb's pessimism about turbulence had been short-sighted. There exist signs that the two issues that concerned Lamb are not disconnected. The connections were brewing for some while, and began to take clearer form recently. This promises renewed vigor in the intellectual pursuit for understanding this long-standing problem. This is not due to some outstanding developments of new tools in the theoretical or experimental study of turbulence *per se*, but rather due to developments in neighboring fields. The great successes of the theory of critical and chaotic phenomena and the popularity of nonlinear physics of classical systems attracted efforts that combined the strength of fields like quantum field theory and condensed matter physics leading to renewed optimism about the solubility of the problem of turbulence. It seems that this area of research will have a renaissance of rapid growth that promises excitement well into the next century.

It is possibly a happy coincidence that hydrodynamic turbulence is considered a problem of immense interest by both physicists and engineers. The physicist tends to appreciate phenomena that display universal characteristics; the engineer may find such characteristics irrelevant since they cannot be manipulated. The engineer seeks control, and control means a ready response to perturbations. Universal phenomena are immune to perturbations. The point is of course that "turbulence" means different things to different researchers. All agree that hydrodynamic turbulence arises in fluids that are highly stressed, or stirred, such that there exist significant fluid velocities (or winds) on the largest scales of motion. The engineer is typically interested in the flow characteristics near the boundaries of the fluid (boundary layers, airplane wings, pipes, turbines etc). By understanding how to manipulate the boundary region one may reduce drag and improve the performance of technological devices. The physicist is interested in the small scale structure of turbulence away from any boundary, where the action of fluid mechanics effectively homogenizes the flow characteristics and where universal phenomena may be sought.

In this context "universal" are those phenomena that are independent of the nature of the fluid (water, oil, honey etc), independent of the mechanism of stirring the flow, and independent of the form of the container of the fluid. They are inherent features of fluid mechanics as a classical field theory. Understanding these universal features can gain only marginal technological improvements. But this is the theoretical challenge that excites the physicist.

To many people turbulence research seems orthogonal to the two main lines of progress in modern physics. On the one hand, tremendous effort has been invested in understanding the structure of matter, with later developments concentrating on ever-diminishing scales of constituent particles using the ever-increasing energies of particle accelerators. On the other hand astronomy and cosmology have exploded with a rich tableau of discoveries at ever-increasing distances from our galaxy. The physics of phenomena on the human scale, phenomena that are of acute interest to the scientist and layman alike, were relegated into a secondary position in the course of the development of the first half of 20th century physics. Of course, problems related to the health and well-being of humans are deservedly being studied in biology and medicine. But physical phenomena that can be observed by simply looking out the window are considered by many as "non-fundamental" and belonging to 19th century research. It is the conviction of the present writers as well as of a growing number of researchers that physics on the human scale offers tremendously rewarding intellectual challenges, some of which were at the core of the recent interest in chaotic phenomena and in the area which is vaguely termed "physics of complex systems". Fluid turbulence, which is the highly complex, chaotic and vortical flow that is characteristic of all fluids under large stresses, is a paramount example of these phenomena that are immensely challenging to the physicist and the mathematician alike. The aim of this paper is to explain why this problem is exciting, why it is difficult, and what are the possible routes that one can traverse in finding the solution. The point of view described here is that of the physicist whose interest is biased in favour of universal phenomena.

II. SOME HISTORY

The mathematical history of fluid mechanics begins with Leonhard Euler who was invited by Frederick the Great to Potsdam in 1741. According to a popular story

(which we could not corroborate) one of his tasks was to engineer a water fountain. As a true theorist, he began by trying to understand the laws of motion of fluids. In 1755 he wrote Newton's laws for a fluid which in modern notation density reads (for the case of constant density) [2]

$$\frac{\partial \boldsymbol{u}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r},t) \cdot \boldsymbol{\nabla} \boldsymbol{u}(\boldsymbol{r},t) = -\boldsymbol{\nabla} p(\boldsymbol{r},t) \ . \qquad (1)$$

Here $\boldsymbol{u}(\boldsymbol{r},t)$ and $p(\boldsymbol{r},t)$ are the fluid velocity and pressure at the spatial point $\boldsymbol{r}$ at time t. The LHS of this "Euler equation" for $\mathbf{u}(\boldsymbol{r},t)$ is just the material time derivative of the momentum, and the RHS is the force, which is represented as the gradient of the pressure imposed on the fluid. In fact, trying to build a fountain on the basis of this equation was bound to fail. This equation predicts, for a given gradient of pressure, velocities that are much higher than anything observed. One missing idea was that of the viscous dissipation that is due to the friction of one parcel of fluid against neighboring ones. The appropriate term was added to (1) by Navier in 1827 by Stokes in 1845 [2]. The result is known as the "Navier-Stokes equations" :

$$\frac{\partial \boldsymbol{u}(\boldsymbol{r},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{r},t) \cdot \boldsymbol{\nabla} \boldsymbol{u}(\boldsymbol{r},t) = -\boldsymbol{\nabla} p(\boldsymbol{r},t) + \nu \nabla^2 \boldsymbol{u}(\boldsymbol{r},t) \ .$$

$$(2)$$

Here ν is the kinematic viscosity, which is about 10^{-2} and 0.15 cm^2/sec for water and air at room temperature respectively. Without the term $\nu \nabla^2 \boldsymbol{u}(\boldsymbol{r},t)$ the kinetic energy $u^2/2$ is conserved; with this term kinetic energy is dissipated and turned into heat. The effect of this term is to stabilize and control the nonlinear energy conserving Euler equation (1).

Straightforward attempts to assess the solutions of this equation may still be very non-realistic. For example, we could estimate the velocity of water flow in any one of the mighty rivers like the Nile or the Volga which drop hundreds of meters in a course of about a thousand kilometers. The typical angle of inclination α is about 10^{-4} radians, and the typical river depth L is about 10 meters. Equating the gravity force αg ($g \simeq 10^3$cm/sec^2) and the viscous drag $\nu d^2 u/dz^2 \sim \nu u/L^2$ we find u to be of the order of 10^7 cm/sec instead of the observed value of about 10^2 cm/sec. This is of course absurd, perhaps to the regret of the white water rafting industry. The resolution of this discrepancy was suggested by Reynolds [3] who stressed the importance of a dimensionless ratio of the nonlinear term to the viscous term in (2). With a velocity drop of the order of U on a scale L the nonlinear term is estimated as U^2/L. The viscous term is about $\nu U/L^2$. The ratio of the two, known as the Reynolds number Re, is UL/ν. The magnitude of Re measures how large is the nonlinearity compared to the effect of the viscous dissipation in a particular fluid flow. For Re$\ll 1$ one can neglect

the nonlinearity and the solutions of the Navier-Stokes equations can be found in close-form in many instances [2]. In natural circumstances Re is very large. For example in the rivers discussed above Re$\simeq 10^7$. Reynolds understood that for Re $\gg 1$ there is no stable stationary solution for the equations of motion. The solutions are strongly affected by the nonlinearity, and the actual flow pattern is complicated, convoluted and vortical. Such flows are called turbulent.

Modern concepts about high Re number turbulence started to evolve with Richardson's insightful contributions [4] which contained the famous "poem" that paraphrased J. Swift: "*Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity -in the molecular sense*". In this way Richardson conveyed an image of the creation of turbulence by large scale forcing, setting up a cascade of energy transfers to smaller and smaller scales by the nonlinearities of fluid motion, until the energy dissipates at small scales by viscosity, turning into heat. This picture led in time to innumerable "cascade models" that tried to capture the statistical physics of turbulence by assuming some thing or other about the cascade process. Indeed, no one in their right mind is interested in the full solution of the turbulent velocity field at all points in space-time. The interest is in the statistical properties of the turbulent flow. Moreover the statistics of the velocity field itself is too heavily dependent on the particular boundary conditions of the flow. Richardson understood that universal properties may be found in the statistics of velocity *differences* $\delta \boldsymbol{u}(\boldsymbol{r}_1, \boldsymbol{r}_2) \equiv \boldsymbol{u}(\boldsymbol{r}_2) - \boldsymbol{u}(\boldsymbol{r}_1)$ across a separation $\boldsymbol{R} = \boldsymbol{r}_2 - \boldsymbol{r}_1$. In taking such a difference we subtract the non-universal large scale motions (known as the "wind" in atmospheric flows). In experiments (see for example [6–9,11]) it is common to consider one dimensional cuts of the velocity field, $\delta u_\ell(R) \equiv \delta \boldsymbol{u}(\boldsymbol{r}_1, \boldsymbol{r}_2) \cdot \boldsymbol{R}/R$. The interest is in the probability distribution function of $\delta u_\ell(R)$ and its moments. These moments are known as the "structure functions" $S_n(R) \equiv \langle \delta u_\ell(R)^n \rangle$ where $\langle \ldots \rangle$ stands for a suitably defined ensemble average. For Gaussian statistics the whole distribution function is governed by the second moment $S_2(R)$, and there is no information to be gained from higher order moments. In contrast, hydrodynamic experiments indicate that turbulent statistics is extremely non-Gaussian, and the higher order moments contain important new information about the distribution functions.

Possibly the most ingenious attempt to understand the statistics of turbulence is due to Kolmogorov who in 1941 [5] proposed the idea of universality (turning the study of small scale turbulence from mechanics to fundamental physics) based on the notion of the "inertial range". The idea is that for very large values of Re there is a wide separation between the "scale of energy input" L and the typical "viscous dissipation scale" η at which viscous friction become important and dumps the energy into heat.

In the stationary situation, when the statistical characteristics of the turbulent flow are time independent, the rate of energy input at large scales (L) is balanced by the rate of energy dissipation at small scales (η), and must be also the same as the flux of energy from larger to smaller scales (denoted $\bar{\epsilon}$) as it is measured at any scale R in the so-called "inertial" interval $\eta \ll R \ll L$. Kolmogorov proposed that the only relevant parameter in the inertial interval is $\bar{\epsilon}$, and that L and η are irrelevant for the statistical characteristics of motions on the scale of R. This assumption means that R is the only available length for the development of dimensional analysis. In addition we have the dimensional parameters $\bar{\epsilon}$ and the mass density of the fluid ρ. From these three parameters we can form combinations $\rho^x \bar{\epsilon}^y R^z$ such that with a proper choice of the exponents x, y, z we form any dimensionality that we want. This leads to detailed predictions about the statistical physics of turbulence. For example, to predict $S_n(R)$ we note that the only combination of $\bar{\epsilon}$ and R that gives the right dimension for S_n is $(\bar{\epsilon}R)^{n/3}$. In particular for $n = 2$ this is the famous Kolmogorov "2/3" law which in Fourier representation is also known as the "-5/3" law. The idea that one extracts universal properties by focusing on statistical quantities can be applied also to the correlations of gradients of the velocity field. An important example is the rate $\epsilon(\mathbf{r}, t)$ at which energy is dissipated into heat due to viscous damping. This rate is roughly $\nu|\nabla\mathbf{u}(\mathbf{r}, t)|^2$. One is interested in the fluctuations of the energy dissipation $\epsilon(\boldsymbol{r}, t)$ about their mean $\bar{\epsilon}$, $\hat{\epsilon}(\mathbf{r}, t) = \epsilon(\mathbf{r}, t) - \bar{\epsilon}$, and how these fluctuations are correlated in space. The answer is given by the often-studied correlation function $K_{\epsilon\epsilon}(R) = \langle\hat{\epsilon}(\mathbf{r} + \mathbf{R}, t)\hat{\epsilon}(\mathbf{r}, t)\rangle$. If the fluctuations at different points were uncorrelated, this function would vanish for all $R \neq 0$. Within the Kolmogorov theory one estimates $K_{\epsilon\epsilon}(R) \simeq \nu^2\bar{\epsilon}^{4/3}R^{-8/3}$, which means that the correlation decays as a power, like $1/R^{8/3}$.

Experimental measurements show that Kolmogorov was remarkably close to the truth. The major aspect of his predictions, i.e. that the statistical quantities depend on the length scale R as power laws is corroborated by experiments. On the other hand, the predicted exponents seem not to be exactly realized. For example, the experimental correlation $K_{\epsilon\epsilon}(R)$ decays according to a power law, $K_{\epsilon\epsilon}(R) \sim R^{-\mu}$ for $\eta \ll R \ll L$, with μ having a numerical value of $0.2 - 0.3$ instead of 8/3 [8]. The structure functions also behave as power laws, $S_n(R) \simeq R^{\zeta_n}$, but the numerical values of ζ_n deviate progressively from $n/3$ when n increases [7,9]. Something fundamental seems to be missing. The uninitiated reader might think that the numerical value of this exponent or another is not a fundamental issue. However one needs to understand that the Kolmogorov theory exhausts the dimensions of the statistical quantities under the assumption that $\bar{\epsilon}$ is the only relevant parameter. Therefore a deviation in the

numerical value of an exponent from the prediction of dimensional analysis requires the appearance of another dimensional parameter. Of course there exists two dimensional parameters, i.e. L and η which may turn out to be relevant. Indeed, experiments indicated that for the statistical quantities mentioned above the energy-input scale L is indeed relevant and it appears as a normalization scale for the deviations from Kolmogorov's predictions: $S_n(R) \simeq (\bar{\epsilon}R)^{n/3}(L/R)^{\delta_n}$ where $\zeta_n = n/3 - \delta_n$. Such form of scaling which deviates from the predictions of dimensional analysis is referred to as "anomalous scaling". The realization that the experimental results for the structure functions were consistent with L rather than η as the normalization scale developed over a long time and involved a large number of experiments; recently the accuracy of determination of the exponents increased appreciably as a result of a clever method of data analysis by Benzi, Ciliberto and coworkers [9]. Similarly a careful demonstration of the appearance of L in the dissipation correlation was achieved by Sreenivasan and coworkers [8]. A direct analysis of scaling exponents ζ_n and μ in a high Reynolds number flow was presented by Praskovskii and Oncley, leading to the same conclusions [10].

III. TURBULENCE AS A FIELD THEORY

Theoretical studies of the universal small scale structure of turbulence can be classified broadly into two main classes. Firstly there is a large body of phenomenological models that by attempting to achieve agreement with experiments reached important insights on the nature of the cascade or the statistics of the turbulent fields [11]. In particular there appeared influential ideas, following Mandelbrot [12], about the fractal geometry of highly turbulent fields which allow scaling properties that are sufficiently complicated to include also non-Kolmogorov scaling. Parisi and Frisch showed that by introducing multifractals one can accomodate the nonlinear dependence of ζ_n and n [13]. However these models are not derived on the basis of the equations of fluid mechanics; one is always left with uncertainties about the validity or relevance of these models. The second class of approaches is based on the equations of fluid mechanics. Typically one acknowledges the fact that fluid mechanics is a (classical) field theory and resorts to field theoretic methods in order to compute statistical quantities. Even though there had been a continuous effort during almost 50 years in this direction, the analytic derivation of the scaling laws for $K_{\epsilon\epsilon}(R)$ and $S_n(R)$ from the Navier-Stokes equations (2) and the calculation of the numerical value of the scaling exponents μ and ζ_n have been among the most elusive goals of theoretical research. Why did it turn out to be so difficult?

To understand the difficulties, we need to elaborate a little on the nature of the field theoretic approach. Suppose that we want to calculate the average response of a turbulent fluid at some point $\mathbf{r}_0$ to forcing at point $\mathbf{r}_1$. The field theoretic approach allows us to consider this response as an infinite sum of all the following processes: firstly there is the direct response at point $\mathbf{r}_0$ due to the forcing at $\mathbf{r}_1$. This response is instantaneous if we assume that the fluid is incompressible (and therefore the speed of sound is infinite). Then there is the process of forcing at $\mathbf{r}_1$ with a response at an intermediate point $\mathbf{r}_2$, which then acts as a forcing for the response at $\mathbf{r}_0$. This intermediate process can take time, and we need to integrate over all the possible positions of point $\mathbf{r}_2$ and all times. This is the second-order term in perturbation theory. Then we can force at $\mathbf{r}_1$, the response at $\mathbf{r}_2$ acting as a forcing for $\mathbf{r}_3$ and the response at $\mathbf{r}_3$ forces a response at $\mathbf{r}_0$. We need to integrate over all possible intermediate positions $\mathbf{r}_2$ and $\mathbf{r}_3$ and all the intermediate times. This is the third-order term in perturbation theory. And so on. The actual response is the infinite sum of all these contributions. In applying this field theoretical method one encounters three main difficulties:

(A) The theory has no small parameter. The usual procedure is to develop the theory perturbatively around the linear part of the equation of motion. In other words, the zeroth order solution of Eq.(2) is obtained by discarding the terms which are quadratic in the velocity field. The expansion parameter is then obtained from the ratio of the quadratic to the linear terms; this ratio is of the order of Reynolds number Re which was defined above. Since we are interested in Re$\gg 1$, naive perturbation expansions are badly divergent. In other words the contribution of the various processes described above increases as $(\mathrm{Re})^n$ with the number n of intermediate points in space-time.

(B) The theory exhibits two types of nonlinear interactions. Both are hidden in the nonlinear term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$ in Eq. (2). The larger of the two is known to any person who watched how a small floating object is entrained in the eddies of a river and swept along a complicated path with the turbulent flow. In a similar way any fluctuation of small scale is swept along by all the larger eddies. Physically this sweeping couples any given scale of motion to all the larger scales. Unfortunately the largest scales contain most of the energy of the flow; these large scale motions are what is experienced as gusts of wind in the atmosphere or the swell in the ocean. In the perturbation theory for $S_n(R)$ one has the consequences of the sweeping effect from all the scale larger than R, with the main contribution coming from the largest, most intensive gusts on the scale of L. As a result these contributions diverge when $L \to \infty$. In the theoretical jargon this is known as "infrared divergences". Such divergences are common in other field theories, with the best known example being quantum electrodynamics. In that

theory the divergences are of similar strength in higher order terms in the series, and they can be removed by introducing finite constants to the theory, like the charge and the mass of the electron. In the hydrodynamic theory the divergences become stronger with the order of the contribution, and to eliminate them in this manner one needs an infinite number of constants. In the jargon such a theory is called "not renormalizable". However, sweeping is just a kinematic effect that does not lead to energy redistribution between scales, and one may hope that if the effect of sweeping is taken care of in a consistent fashion a renormalizable theory might emerge. This redistribution of energy results from the second type of interaction, that stems from the shear and torsion effects that are sizable only if they couple fluid motions of comparable scales. The second type of nonlinearity is smaller in size but crucial in consequence, and it may certainly lead to a scale-invariant theory.

(C) Nonlocality of interaction in $\mathbf{r}$ space. One recognizes that the gradient of the pressure is dimensionally the same as $(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}$, and the fluctuations in the pressure are quadratic in the fluctuations of the velocity. This means that the pressure term is also nonlinear in the velocity. However, the pressure at any given point is determined by the velocity field everywhere. Theoretically one sees this effect by taking the divergence of Eq.(2). This leads to the equation $\nabla^2 p = \boldsymbol{\nabla} \cdot [(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}]$. The inversion of the Laplacian operator involves an integral over all space. Physically this stems from the fact that in the incompressible limit of the Navier-Stokes equations sound speed is infinite and velocity fluctuations in all distant points are instantaneously coupled.

Indeed, these difficulties seemed to complicate the application of field theoretic methods to such a degree that a wide-spread feeling appeared to the effect that it is impossible to gain valuable insight into the universal properties of turbulence along these lines, even though they proved so fruitful in other field theories. The present authors (as well as other researchers starting with Kraichnan [] and recently Migdal [14], Polyakov [15], Eyink [16] etc.) think differently, and in the rest of this paper we will explain why.

The first task of a successful theory of turbulence is to overcome the existence of the interwoven nonlinear effects that were explained in difficulty (B). This is not achieved by directly applying a formal field-theoretical tool to the Navier-Stokes equations. It does not matter whether one uses standard field theoretic perturbation theory [17], path integral formulation, renormalization group [18] ϵ-expansion, large N-limit [19] or one's formal method of choice. One needs to take care of the particular nature of hydrodynamic turbulence as embodied in difficulty (B) *first*, and *then* proceed using formal tools.

The removal of the effects of sweeping is based on Richardson's remark that universality in turbulence is expected for the statistics of velocity *differences across a*

length scale R rather than for the statistics of the velocity field itself. The velocity fields are dominated by the large scale motions that are not universal since they are produced directly by the agent that forces the flow. This forcing agent differs in different flow realizations (atmosphere, wind tunnels, channel flow etc.). Richardson's insight was developed by Kraichnan who attempted to cast the field theoretic approach in terms of Lagrangian paths, meaning a description of the fluid flow which follows the paths of every individual fluid particle. Such a description automatically removes the large scale contributions [20]. Kraichnan's approach was fundamentally correct, and gave rise to important and influential insights in the description of turbulence, but did not provide a convenient technical way to consider all the orders of perturbation theory. The theory does not provide transparent rules how to consider an arbitrarily high term in the perturbation theory. Only low order truncations were considered.

A way to overcome difficulty (B) was suggested by Belinicher and L'vov [21] who introduced a novel transformation that allowed on one hand the elimination of the sweeping that leads to infrared divergences, and on the other hand allows the development of simple rules for writing down any arbitrary order in the perturbation theory for the statistical quantities. The essential idea in this transformation is the use of a coordinate frame in which velocities are measured relative to the velocity of *one* fluid particle. The use of this transformation allowed the examination of the structure functions of velocity differences $S_n(R)$ to all orders in perturbation theory. Of course, difficulty (A) remains; the perturbation series still diverges rapidly for large values of Re, but now standard field theoretic methods can be used to reformulate the perturbation expansion such that the viscosity is changed by an effective "eddie viscosity". The theoretical tool that achieves this exchange is known in quantum field theory as the Dyson line resummation [23]. The result of this procedure is that the effective expansion parameter is no longer Re but an expansion parameter of the order of unity. Of course, such a perturbation series may still diverge as a whole. Notwithstanding it is crucial to examine first the order-by-order properties of series of this type.

Such an examination leads to a major surprise: every term in this perturbation theory remains finite when the energy-input scale L goes to ∞ and the viscous-dissipation scale η goes to 0 [23]. The meaning of this is that the perturbative theory for S_n does not indicate the existence of any typical length-scale. Such a length is needed in order to represent deviations in the scaling exponents from the predictions of Kolmogorov's dimensional analysis in which both scales L and η are assumed irrelevant. In other areas of theoretical physics in which anomalous scaling has been found it is common that already the perturbative series indicates this phenomenon.

In many cases this is seen in the appearance of logarithmic divergences that must be tamed by truncating the integrals at some renormalization length. Hydrodynamic turbulence seems at this point different. The nonlinear Belinicher-L'vov transformation changes the underlying linear theory such that the resulting perturbative scheme for the structure functions is finite order by order [21,23]. The physical meaning of this result is that as much as can be seen from this perturbative series the main effects on the statistical quantities for velocity differences across a scale R come from activities of scales comparable to R. This is the perturbative justification of the Richardson-Kolmogorov cascade picture in which widely separated scales do not interact.

Consequently the main question still remains: how does a renormalization scale appear in the statistical theory of turbulence?

It turns out that there are two different mechanisms that furnish a renormalization scale, and that finally *both L and η* appear in the theory. The viscous scale η appears via a rather standard mechanism that can be seen in perturbation theory as logarithmic divergences, but in order to see it one needs to consider the statistics of gradient fields rather than the velocity differences themselves [27,24]. For example, considering the perturbative series for $K_{\epsilon\epsilon}(R)$, which is the correlation function of the rate of energy dissipation $\nu|\boldsymbol{\nabla}\boldsymbol{u}|^2$, leads immediately to the discovery of logarithmic ultraviolet divergences in every order of the perturbation theory. These divergences are controlled by an ultraviolet cutoff scale which is identified as the viscous-dissipation scale η acting here as the renormalization scale. The summation of the infinite series results in a factor $(R/\eta)^{2\Delta}$ with some anomalous exponent Δ which is, generally speaking, of the order of unity. The appearance of such a factor means that the actual correlation of two R-separated dissipation fields is much larger, when R is much larger than η, than the naive prediction of dimensional analysis. The physical explanation of this renormalization [27,28] is the effect of the multi-step interaction of two R-separated small eddies of scale η with a large eddy of scale R via an infinite set of eddies of intermediate scales. The net result on the scaling exponent is that the exponent μ changes from $8/3$ as expected in the Kolmogorov theory to $8/3 - 2\Delta$.

At this point it is important to understand what is the numerical value of the anomalous exponent Δ. In [24] there was found an exact sum that forces a relation between the numerical value of Δ and the numerical value of the exponent ζ_2 of $S_2(R)$, $\Delta = 2 - \zeta_2$. Such a relation between different exponents is known in the jargon as a "scaling relation" or a "bridge relation". Physically this relation is a consequence of the existence of a universal nonequilibrium stationary state that supports an energy flux from large to small scales [24,25]. The scaling relation for Δ has far-reaching implications for the theory of the structure functions. It was explained that with this

value of Δ the series for the structure functions $S_n(R)$ diverge when the energy-input scale L approaches ∞ as powers of L, like $(L/R)^{\delta_n}$. The anomalous exponents δ_n are the deviations of the exponents of $S_n(R)$ from their Kolmogorov value. This is a very delicate and important point, and we therefore expand on it. Think about the series representation of $S_n(R)$ in terms of lower order quantities, and imagine that one succeeded to resum it into an operator equation for $S_n(R)$. Typically such a resummed equation may look like $[1 - \hat{O}]S_n(R)] =$RHS, where $\hat{O}$ is some integro-differential operator which is not small compared to unity. If we expand this equation in powers of $\hat{O}$ around the RHS we regain the infinite perturbative series that we started with. However, now we realize that the equation possesses also homogeneous solutions, solutions of $[1 - \hat{O}]S_n(R)] = 0$ which are inherently nonperturbative since they can no longer be expanded around a RHS. These homogeneous solutions may be much larger than the inhomogeneous perturbative solutions. Of course, homogeneous solutions must be matched with the boundary conditions at $R = L$, and this is the way that the energy input scale L appears in the theory. This is particularly important when the homogeneous solution diverge in size when $L \to \infty$ as is indeed the case for the problem at hand.

The next step in the theoretical development is to understand how to compute the anomalous exponents δ_n. The divergence of the perturbation theory for $S_n(R)$ with $L \to \infty$ forces us to seek a nonperturbative handle on the theory. One finds this in the idea that there exists always a global balance between energy input and dissipation, which may be turned into a nonperturbative constraint on each n-th order structure function [25]. Using the Navier-Stokes equations (2) one derives the set of equations of motion

$$\frac{\partial S_n(R,t)}{\partial t} + D_n(R,t) = \nu J_n(R,t) , \qquad (3)$$

where D_n and J_n stem from the nonlinear and the viscous terms in (2) respectively. To understand the physical meaning of this equation note that $S_2(R)$ is precisely the mean kinetic energy of motions of size R. The term $D_2(R)$ whose meaning is the rate of energy flux through the scale R is known exactly: $D_2(R) = dS_3(R)/dR$. The term $\nu J_2(R)$ is precisely the rate of energy dissipation due to viscous effects. The higher order equation for $n > 2$ are direct generalizations of this to higher order moments. In the stationary state the time derivative vanishes and one has the balance equation $D_n(R) = \nu J_n(R)$. For $n = 2$ it reflects the balance between energy flux and energy dissipation. The evaluation of $D_n(R)$ for $n > 2$ requires dealing with the difficulty (C) of the nonlocality of the interaction, but it does not pose conceptual difficulties. It was shown [25] that $D_n(R)$ is of the order of dS_{n+1}/dR. On the other hand, the evaluation of $J_n(R)$ raises a number of very interesting issues whose resolu-

tion lies at the heart of the universal scaling properties of turbulence. Presently not all of these issues have been resolved, and we briefly mention here some ground on which progress has been made by the present authors.

From the derivation of Eq.(3) one finds that $J_n(R)$ consists of a correlation of $\nabla^2 u$ with $n-2$ velocity differences across a scale $R = |\boldsymbol{r}_1 - \boldsymbol{r}_2)|$: $\langle \nabla^2 u(\boldsymbol{r}_1)[\delta u_\ell(\boldsymbol{r}_1, \boldsymbol{r}_2)]^{n-2} \rangle$. The question is how to evaluate such a quantity in terms of the usual structure functions $S_n(R)$. Recall that a gradient of a field is the difference in the field values at two points divided by the separation when the latter goes to zero. In going to zero one necessarily crosses the dissipative scale. To understand what happens in this process one needs first to introduce many-point correlation functions of a product of n velocity differences:

$$F_n(\boldsymbol{r}_0|\boldsymbol{r}_1, \dots \boldsymbol{r}_n) \equiv \langle \delta u_\ell(\boldsymbol{r}_0, \boldsymbol{r}_1) \dots \delta u_\ell(\boldsymbol{r}_0, \boldsymbol{r}_n) \rangle . \quad (4)$$

Next we need to formulate rules for the evaluation of such correlation functions of velocity differences when some of the coordinates get very close to each other. For example, a gradient $\partial/\partial r_\alpha$ can be formed from the limit $\boldsymbol{r}_1 \to \boldsymbol{r}_0$ when we divide by $r_{1,\alpha} - r_{0,\alpha}$. These rules are known in the theoretical jargon as "fusion rules". The fusion rules for hydrodynamic turbulence were presented in [29]. They show that when p coordinates in F_n are separated by a small distance r, and the remaining $n-p$ coordinates are separated by a large distance R, then the scaling dependence on r is like that of $S_p(r)$, i.e. r^{ζ_p}. This is true until r crosses the dissipative scale. Assuming that below the viscous-dissipation scale η derivatives exist and the fields are smooth, one can estimate gradients at the end ofthe smooth range by dividing differences across η by η. The question is, what is the appropriate cross-over scale to smooth behaviour? Is there just one cross-over scale η, or is there a multiplicity of such scales, depending on the function one is studying? For example, when does the above n-point correlator become differentiable as a function of r when p of its coordinates approach $\boldsymbol{r}_0$? Is that typical scale the same as the one exhibited by $S_p(r)$ itself, or does it depend on p and n and on the remaining distances of the remaining $n-p$ coordinates that are still far away from $\boldsymbol{r}_0$?

The answer is that there is a multiplicity of cross-over scales. For the n-point correlator discussed above we denote the dissipative scale as $\eta(p, n, R)$, and it depends on each of its arguments [29,26]. In particular it depends on the inertial range variables R and this dependence must be known when one attempts to determine the scaling exponents ζ_n of the structure functions. In brief, this line of thought leads to a set of non-trivial scaling relations. For example we confirm the phenomenologically conjectured [11] "bridge relation" $\mu = 2 - \zeta_6$ (in close agreement with the experimental values) and predict that although the ζ_n are not Kolmogorov, they are nevertheless linear in n for large n.

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IV. SUMMARY

It appears that there are four conceptual steps in the construction of a theory of the universal anomalous statistics of turbulence on the basis of the Navier-Stokes equations. First one needs to take care of the sweeping interactions that mask the scale invariant theory [21,23]. After doing so the perturbation expansion converges order by order, and the Kolmogorov scaling of the velocity structure functions is found as a perturbative solution. Secondly one understands the appearance of the viscous-dissipation scale η as the natural normalization scale in the theory of the correlation functions of the gradient fields [27,24]. This step is similar to critical phenomena and it leads to a similarly rich theory of anomalous behaviour of the gradient fields. Only the tip of the iceberg was considered above. In fact when one considers the correlations of tensor fields which are constructed from $\partial u_\alpha / \partial r_\beta$ (rather than the scalar field ϵ) one finds that every field with a different transformation property under the rotation of the coordinates has its own independent scaling exponent which is the analog of Δ above [30]. The third step is the understanding of the divergence of the diagrammatic series for the structure functions as a whole [25]. This sheds light on the emergence of the energy-input scale L as a normalization length in the theory of turbulence. This means that the Kolmogorov basic assertion that there is no typical scale in the expressions for statistical quantities involving correlations across a scale R when $\eta \ll R \ll L$ is doubly wrong. In general both lengths appear in dimensionless combinations and change the exponents from the predictions of dimensional analysis. Examples of correlation functions in which both normalization scales L and η appear simultaneously were given explicitly [30]. Last but not least is the formulation of the fusion rules and the exposition of the multiplicity of the dissipative scales which should eventually result in a satisfactory description of all the scaling properties [26].

The road ahead is not fully charted, but it seems that some of the conceptual difficulties have been surmounted. We believe that the crucial building blocks of the theory are now available, and they begin to delineate the structure of the theory. We hope that the remaining 4 years of this century will suffice to achieve a proper understanding of the anomalous scaling exponents in turbulence. Considerable work, however, is still needed in order to fully clarify many aspects of the problem, and most of them are as exciting and important as the scaling properties. There are universal aspects that go beyond exponents, such as distribution functions and the eddy viscosity, and there are important non-universal aspects like the role of inhomogeneities, the effect of boundaries and so on. Progress on these issues will bring the theory closer to the concern of the engineers. The marriage of physics and engineering will be the challenge of the 21st century.

[1] S. Goldstein, Ann. Rev. Fluid Mech, **1**, 23 (1969)
[2] H. Lamb *Hydrodynamics*, (Dover publications, 1945)
[3] O. Reynolds, Phil. Trans. Roy. Soc., London **A186**, 123 (1894).
[4] L.F. Richardson, *Weather Prediction by Numerical Process*, (Cambridge, 1922)
[5] A.N. Kolmogorov, Dokl. Acad. Nauk SSSR, **30**, 9 (1941).
[6] A. S. Monin and A. M. Yaglom. *Statistical Fluid Mechanics: Mechanics of Turbulence*, volume II. The MIT Press, Cambridge, Mass., 1973.
[7] F. Anselmet, Y. Gagne, E.J. Hopfinger, and R.A. Antonia, J. Fluid Mech. **140**, 63 (1984).
[8] K.R. Sreenivasan and P. Kailasnath. Phys. Fluids A **5**, 512, (1993).
[9] R. Benzi, S. Ciliberto, R. Tripiccione, C. Baudet, F. Massaioli and S. Succi, Phys. Rev. E **48**, R29 (1993).
[10] A. Praskovskii and S. Oncley, Phys. Fluids, **6**,2886 (1994)
[11] Uriel Frisch. *Turbulence: The Legacy of A.N. Kolmogorov*. Cambridge University Press, Cambridge, 1995.
[12] B.B. Mandelbrot, J. Fluid Mech. **62**,331 (1974)
[13] G. Parisi and U. Frisch in "Turbulence and Predictability in Geophyiscal Fluid Dynamics, eds. M. Ghil, R. Benzi and G. Parisi, (North Holland, Amsterdam 1985)
[14] A.A. Migdal, Int. J. of Mod. Phys. A **9**, 1197 (1994)
[15] A. M. Polyakov, Nucl. Phys. B **396**, 367 (1993)
[16] G.L. Eyink, Phys.Lett. A **172**, 355 (1993)
[17] H. W. Wyld. Ann. Phys., **14**, 143 (1961).
[18] V. Yakhot and S.A. Orszag, Phys.Rev. Lett **57**,1722 (1986)
[19] C-Y. Mou and P. B. Weichman. Phys.Rev E **52**, 3738 (1995)
[20] R.H. Kraichnan, Phys. Fluids, **8**, 575 (1965)
[21] V. I. Belinicher and V. S. L'vov. Sov. Phys. JETP, **66**, 303 (1987).

[22] V.S. L'vov and I. Procaccia "Exact Resummation in the Theory of Hydrodynamic Turbulence": 0. Line Resummed Diagrammatic Perturbation Approach" in F. David P. Ginsparg, and J. Zinn-Justin, eds. Les Houches session LXII, 1994, "Fluctuating Geometries in Statistical Mechanics and Field Theory" (Elsevier, 1995).

[23] V.S. L'vov and I. Procaccia "Exact Resummation in the Theory of Hydrodynamic Turbulence": I. The Ball of Locality and Normal Scaling",Phys. Rev. E, 52, 3840 (1995)

[24] V.S. L'vov and I. Procaccia "Exact Resummation in the Theory of Hydrodynamic Turbulence": II. The Ladder to Anomalous Scaling",Phys. Rev. E, 52, 3858 (1995)

[25] V.S. L'vov and I. Procaccia "Exact Resummation in the Theory of Hydrodynamic Turbulence": III. Scenarios for Multiscaling and Intermittency". Phys.Rev. E, in press

[26] V.S. L'vov and I. Procaccia "Exact Resummation in the Theory of Hydrodynamic Turbulence": IV. Fusion Rules and the Multiplicity of Viscous Scales", Phys.Rev E, to be published.

[27] V.S. L'vov and V. Lebedev, Europhys. Lett. **29**, 681 (1995)

[28] V.S. L'vov and I. Procaccia Phys.Rev. Lett. **74**, 2690 (1995) .

[29] V.S. L'vov and I. Procaccia, "Fusion Rules in Turbulent Systems with Flux Equilibrium", Phys. Rev. Lett. Submitted.

[30] V.S. L'vov, E. Podivilov and I. Procaccia, Phys. Rev. Lett. Submitted.